

A Detailed Analysis of a Vibrating String

May 2025

Abstract

In this work, we explore the spatial and temporal distribution of energy in standing waves formed on elastic strings, with emphasis on the localization of kinetic energy at antinodes and potential energy at nodes. We begin by analyzing the behavior of ideal standing waves, where the total energy is periodically exchanged between kinetic and potential forms without net transport. This serves as a foundation for discussing the limitations of the linear wave approximation, which assumes small amplitude oscillations and constant tension. We then investigate conditions under which this approximation fails, such as in high-amplitude or high-frequency regimes, where nonlinear effects become significant. These include curvature-dependent tension variation, dispersion-induced pulse narrowing, and energy transfer between vibrational modes, ultimately leading to the generation of higher harmonics.

Detailed Analysis of the Standing Wave on a String

Wave Equation

The wave equation describing the displacement $y(x, t)$ of a point on a string fixed at both ends, forming a standing wave, is given by:

$$y(x, t) = 2A \sin(kx) \sin(\omega t), \quad (1)$$

where:

- A is the amplitude of the individual traveling waves,
- $k = \frac{2\pi}{\lambda}$ is the wave number,
- $\omega = 2\pi f$ is the angular frequency,
- λ is the wavelength,
- f is the frequency.

At the initial time $t = 0$, the displacement and velocity of the string points are:

$$y(x, 0) = 0, \quad \text{and} \quad \frac{\partial y}{\partial t}(x, 0) = 2\omega A \sin(kx). \quad (2)$$

Kinetic Energy of an Infinitesimal String Element

Consider a small element of the string of length dx located at position x . This element has a mass:

$$dm = \mu dx, \quad (3)$$

where μ is the linear mass density of the string (mass per unit length).

The kinetic energy dK of this small element is:

$$dK = \frac{1}{2}dm \left(\frac{\partial y}{\partial t} \right)^2 = \frac{1}{2}\mu dx \left(\frac{\partial y}{\partial t} \right)^2. \quad (4)$$

The velocity of each point on the string as a function of position and time is obtained by differentiating the displacement with respect to time:

$$\frac{\partial y}{\partial t}(x, t) = 2\omega A \sin(kx) \cos(\omega t). \quad (5)$$

Therefore, the kinetic energy density (kinetic energy per unit length) is:

$$\frac{dK}{dx} = \frac{1}{2}\mu (2\omega A \sin(kx) \cos(\omega t))^2 = 2\mu\omega^2 A^2 \sin^2(kx) \cos^2(\omega t). \quad (6)$$

Potential Energy of the String Element

The potential energy arises due to the elastic deformation of the string. The force responsible for this is the tension T in the string.

The potential energy dU of the element depends on the elongation dS caused by the deformation:

$$dU = T dS. \quad (7)$$

The change in length dS of the string element relative to the unstretched length dx is:

$$dS = \sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} dx - dx = \left(\sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} - 1 \right) dx.$$

Thus, the potential energy density is:

$$\frac{dU}{dx} = T \left(\sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} - 1 \right). \quad (8)$$

Since

$$\frac{\partial y}{\partial x}(x, t) = 2kA \cos(kx) \sin(\omega t), \quad (9)$$

we have

$$\frac{dU}{dx} = T \left(\sqrt{1 + 4k^2 A^2 \cos^2(kx) \sin^2(\omega t)} - 1 \right). \quad (10)$$

Behavior at Nodes and Antinodes

The nodes (points of zero displacement) occur at positions where $x = 0, \frac{\lambda}{2}, \lambda, \dots$. At these points:

$$\sin(kx) = 0,$$

so the kinetic energy density at the node is zero:

$$\left. \frac{dK}{dx} \right|_{x=0} = 0, \quad (11)$$

while the potential energy density is:

$$\left. \frac{dU}{dx} \right|_{x=0} = T \left(\sqrt{1 + 4k^2 A^2 \sin^2(\omega t)} - 1 \right). \quad (12)$$

At $t = 0$, the potential energy at the node is zero, and at $t = \frac{T}{4}$ (where $T = \frac{2\pi}{\omega}$ is the period), the potential energy is maximal.

In contrast, at antinodes (positions of maximum displacement), for example $x = \frac{\lambda}{4}$, the situation reverses:

$$\left. \frac{dK}{dx} \right|_{x=\frac{\lambda}{4}} = 2\mu\omega^2 A^2 \cos^2(\omega t), \quad (13)$$

and

$$\left. \frac{dU}{dx} \right|_{x=\frac{\lambda}{4}} = 0. \quad (14)$$

This means the energy oscillates between kinetic and potential forms locally, with energy flowing back and forth between nodes and antinodes during a cycle.

Total Kinetic Energy of the String

To find the total kinetic energy K of the string of length L , integrate the kinetic energy density over the entire string:

$$K = \int_0^L \frac{dK}{dx} dx = \int_0^L 2\mu\omega^2 A^2 \sin^2(kx) \cos^2(\omega t) dx \quad (15)$$

$$= 2\mu\omega^2 A^2 \cos^2(\omega t) \int_0^L \sin^2(kx) dx. \quad (16)$$

Using the integral

$$\int \sin^2(ax) dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C,$$

we get

$$\int_0^L \sin^2(kx) dx = \frac{L}{2} - \frac{\sin(2kL)}{4k}. \quad (17)$$

Therefore,

$$K = 2\mu\omega^2 A^2 \cos^2(\omega t) \left(\frac{L}{2} - \frac{\sin(2kL)}{4k} \right). \quad (18)$$

Total Potential Energy of the String

Similarly, for the total potential energy U , using the approximation for small slopes:

$$\sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} \approx 1 + \frac{1}{2} \left(\frac{\partial y}{\partial x} \right)^2,$$

we have

$$dU \approx \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 dx. \quad (19)$$

Thus,

$$U = \int_0^L \frac{dU}{dx} dx \approx \frac{1}{2} T \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx \quad (20)$$

$$= \frac{1}{2} T \int_0^L (2kA \cos(kx) \sin(\omega t))^2 dx \quad (21)$$

$$= 2Tk^2 A^2 \sin^2(\omega t) \int_0^L \cos^2(kx) dx. \quad (22)$$

Using

$$\int \cos^2(ax) dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C,$$

we get

$$\int_0^L \cos^2(kx) dx = \frac{L}{2} + \frac{\sin(2kL)}{4k}. \quad (23)$$

Therefore,

$$U = 2Tk^2 A^2 \sin^2(\omega t) \left(\frac{L}{2} + \frac{\sin(2kL)}{4k} \right). \quad (24)$$

Condition for Standing Waves

Since the string is fixed at both ends, its length L must be an integer multiple of half wavelengths:

$$L = n \frac{\lambda}{2}, \quad n = 1, 2, 3, \dots \quad (25)$$

In this case,

$$\sin(2kL) = \sin\left(2k \cdot n \frac{\lambda}{2}\right) = \sin(2\pi n) = 0.$$

Hence, the total kinetic and potential energies simplify to:

$$K(t) = \mu L \omega^2 A^2 \cos^2(\omega t), \quad (26)$$

$$U(t) = \mu L \omega^2 A^2 \sin^2(\omega t). \quad (27)$$

Total Energy and Its Conservation

Summing the total kinetic and potential energies:

$$E_{\text{total}} = K(t) + U(t) = \mu L \omega^2 A^2 (\sin^2(\omega t) + \cos^2(\omega t)) = \mu L \omega^2 A^2, \quad (28)$$

which is constant in time, reflecting the conservation of energy in the standing wave system.

Alternatively, the total energy can be expressed as:

$$E_{\text{total}} = \frac{1}{2} \mu L \omega^2 (2A)^2, \quad (29)$$

since the amplitude of the standing wave is $2A$.

Comparison with Simple Harmonic Oscillator

The form of the total energy E_{total} is analogous to the energy of a simple harmonic oscillator (SHO), $E = \frac{1}{2} D A^2$, where D is the effective stiffness.

Comparing, we find:

$$D = 2\mu L \omega^2 = \frac{2\pi^2 T L}{\lambda^2}. \quad (30)$$

This parameter D depends on the tension T , the string length L , and the wavelength λ . Physically, it represents the "stiffness" of the string analogous to a spring constant. For shorter wavelengths, the string behaves "stiffer."

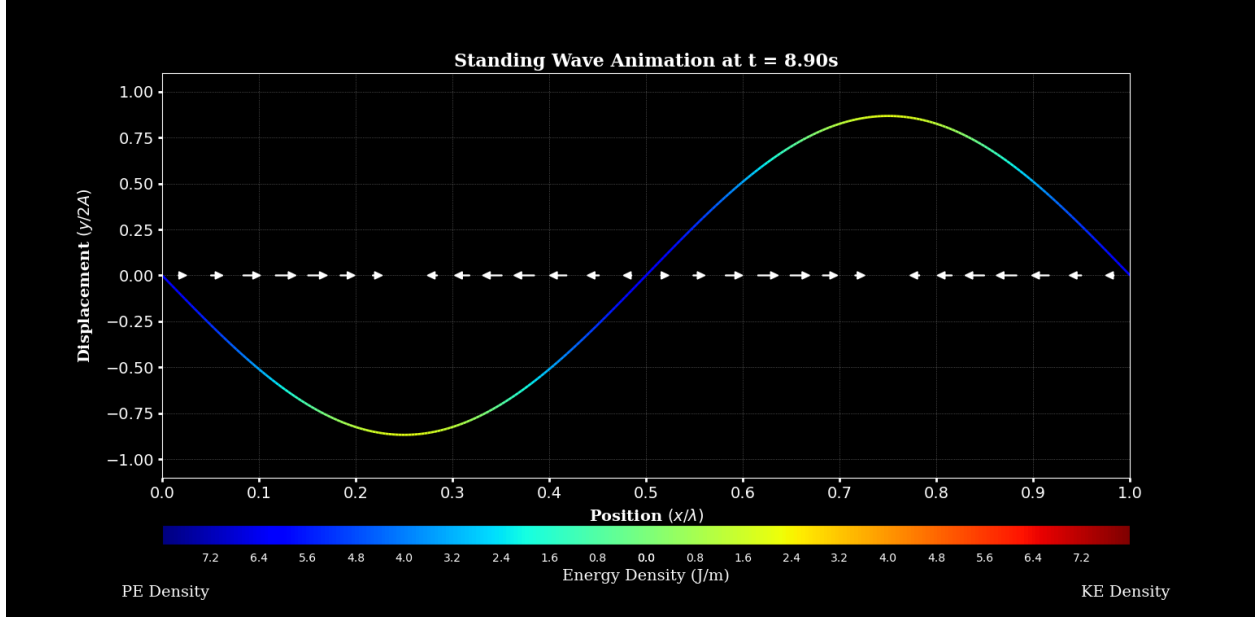


Figure 1: Standing wave, arrows show direction and magnitude of energy

Energy Flow in the String

The instantaneous energy flux F at position x is given by the negative time derivative of the total energy contained from 0 to x :

$$F = -\frac{\partial}{\partial t} (K + U) \quad (31)$$

$$F = -T\kappa^2 A^2 \frac{\partial}{\partial t} \left[\left(x - \frac{\sin(2\kappa x)}{2\kappa} \right) \cos^2(\omega t) + \left(x + \frac{\sin(2\kappa x)}{2\kappa} \right) \sin^2(\omega t) \right] \quad (32)$$

After detailed calculation (omitted here for brevity), one obtains:

$$F = -\omega T \kappa A^2 \sin(2\kappa x) \sin(2\omega t). \quad (33)$$

Average Energy Flow Over One Period

Averaging the energy flow over one period $T_o = \frac{2\pi}{\omega}$:

$$\langle F \rangle = \frac{1}{T_o} \int_0^{T_o} F dt = -\frac{\omega T \kappa A^2 \sin(2\kappa x)}{T_o} \int_0^{T_o} \sin(2\omega t) dt = 0. \quad (34)$$

Thus, the average energy flow at any point on the string is zero, confirming that in a standing wave there is no net transfer of energy along the string, but rather a periodic local exchange of energy between kinetic and potential forms, moving between nodes and antinodes.

Quantization of Total Energy

The total energy stored in a standing wave of length L is:

$$E_{\text{total}} = TLk^2 A^2 \quad (35)$$

For a half-wavelength segment $L = \frac{\lambda}{2}$, and $k = \frac{2\pi}{\lambda}$, we get:

$$E_{\lambda/2} = T \cdot \frac{\lambda}{2} \cdot \left(\frac{2\pi}{\lambda}\right)^2 A^2 = \frac{2T(\pi A)^2}{\lambda} \quad (36)$$

Hence, the energy of a standing wave with n antinodes is:

$$E_{\text{total}} = n \cdot E_{\lambda/2} = n \cdot \frac{2T(\pi A)^2}{\lambda} \quad (37)$$

This shows that the energy is **quantized** in units of half-wavelength segments.

Energy Loss Due to Damping

Suppose a damping force per unit length acts on the string, proportional to the local velocity and in the opposite direction:

$$\Phi(x, t) = -\frac{b}{L} \dot{y}(x, t) \quad (38)$$

To find the work done by this damping force per unit length over one period, we integrate over the displacement dy at a given point:

$$W_{\text{diss}}(x) = \int_0^{2A \sin(kx)} \Phi(x, \dot{y}) dy = - \int_0^{2A \sin(kx)} \frac{b}{L} \dot{y} dy \quad (39)$$

Now, integrating over the full string length to get the total energy E_α lost in one period:

$$E_\alpha = \int_0^L \int_0^{2A \sin(kx)} \Phi(x, \dot{y}) dy dx \quad (40)$$

$$E_\alpha = -\frac{b}{L} \int_0^L \int_0^{2A \sin(kx)} \left(\frac{dy}{dt}\right)^2 dy dx \quad (41)$$

Since:

$$\dot{y}(x, t) = -2A\omega \sin(kx) \sin(\omega t) \quad (42)$$

we square it and average over one period T_0 :

$$\frac{E_\alpha}{4} = \frac{b}{L} \int_0^L \int_0^{T_0/4} (4A^2\omega^2 \sin^2(kx) \sin^2(\omega t)) dt dx \quad (43)$$

$$\frac{E_\alpha}{4} = \frac{4bA^2\omega^2}{L} \cdot \int_0^L \sin^2(kx) dx \cdot \int_0^{T_0/4} \sin^2(\omega t) dt \quad (44)$$

Solving the integrals:

$$\int_0^L \sin^2(kx) dx = \frac{L}{2}, \quad \int_0^{T_0/4} \sin^2(\omega t) dt = \frac{T_0}{8} \quad (45)$$

Therefore:

$$\frac{E_\alpha}{4} = \frac{4bA^2\omega^2}{L} \cdot \frac{L}{2} \cdot \frac{T_0}{8} \Rightarrow E_\alpha = bT_0\omega^2 A^2 = 2\pi b\omega A^2 \quad (46)$$

Quality Factor

The quality factor Q of the system is defined as the ratio of total energy stored to energy lost per cycle, multiplied by 2π :

$$Q = 2\pi \cdot \frac{E_{\text{total}}}{E_\alpha} = 2\pi \cdot \frac{TLk^2 A^2}{2\pi b\omega A^2} = \frac{\mu L\omega}{b} \quad (47)$$

To ensure the validity of the weak damping approximation, we require:

$$b \ll \mu L\omega \quad (48)$$

This condition ensures that energy loss is small compared to the total energy stored in the system, preserving the nature of the standing wave.

Nonlinear Wave Equation for Large Amplitude Oscillations

If the amplitude of oscillation is not small compared to the wavelength, the classical linear wave equation is no longer valid. Instead, the governing equation becomes:

$$\frac{\partial}{\partial x} \left(T(x, t) \frac{\partial y}{\partial x} \right) - b \frac{\partial y}{\partial t} - F \frac{\partial^2 \kappa}{\partial x^2} = \mu(x) \frac{\partial^2 y}{\partial t^2} \quad (49)$$

where $T(x, t)$ is the tension in the string, which now depends on the local elongation.

A reasonable assumption is that the tension depends on the elongation of the string segment:

$$T(x, t) = T_c + \alpha \left(\sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} - 1 \right) \quad (50)$$

where T_c is the baseline tension and α is a proportionality constant reflecting tension increase due to elongation.

Under this assumption, the partial differential equation (PDE) reduces to:

$$\frac{\partial T}{\partial x} \frac{\partial y}{\partial x} + T(x, t) \frac{\partial^2 y}{\partial x^2} - b \frac{\partial y}{\partial t} = \mu(x) \frac{\partial^2 y}{\partial t^2} \quad (51)$$

This nonlinear PDE is analytically intractable, so we proceed with numerical methods. Specifically, we use the Finite Element Method (FEM) for spatial discretization and the Newmark method for temporal integration.

Weak Formulation

Multiplying both sides by a test function $u(x)$ in the solution space and integrating over the string length L , we transform the strong form into its weak form:

$$\int_0^L \frac{\partial T}{\partial x} \frac{\partial y}{\partial x} u(x) dx + \int_0^L T(x, t) \frac{\partial^2 y}{\partial x^2} u(x) dx - b \int_0^L \frac{\partial y}{\partial t} u(x) dx = \int_0^L \mu(x) \frac{\partial^2 y}{\partial t^2} u(x) dx \quad (52)$$

Integration by parts and boundary term evaluations yield:

$$\int_0^L \mu(x) \frac{\partial^2 y}{\partial t^2} u(x) dx + \int_0^L b \frac{\partial y}{\partial t} u(x) dx - \int_0^L y(x, t) \frac{\partial T}{\partial x} \frac{\partial u}{\partial x} dx = 0 \quad (53)$$

Finite Element Approximation

We approximate the solution as a sum over basis functions $\phi_i(x)$:

$$y(x, t) = \sum_{i=1}^N U_i(t) \phi_i(x) \quad (54)$$

where the basis functions are piecewise linear (rooftop functions):

$$\phi_i(x) = \begin{cases} 1 + \frac{x-x_i}{x_i-x_{i-1}}, & x_{i-1} \leq x \leq x_i \\ 1 - \frac{x-x_i}{x_{i+1}-x_i}, & x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad \text{for } 1 \leq i \leq N \quad (55)$$

with $x_0 = 0$ and $x_{N+1} = L$.

The derivatives become:

$$\frac{\partial y}{\partial x} = \sum_{i=1}^N U_i(t) \frac{\partial \phi_i}{\partial x}, \quad \frac{\partial^2 y}{\partial x^2} = \sum_{i=1}^N U_i(t) \frac{\partial^2 \phi_i}{\partial x^2} = 0, \quad (56)$$

since ϕ_i are piecewise linear and their second derivative vanishes.

The tension in a small element of length Δx is:

$$T(x, t) = T_c + \alpha \left(\sqrt{1 + \left(\sum_{i=1}^N U_i(t) \frac{\partial \phi_i}{\partial x} \right)^2} - 1 \right) \Delta x \quad (57)$$

Assuming the test functions equal the basis functions (Galerkin method), the weak form leads to the system of ODEs:

$$\sum_{i=1}^N \ddot{U}_i(t) M_{ji} + \sum_{i=1}^N \dot{U}_i(t) R_{ji} + \sum_{i=1}^N U_i(t) K_{ji} = 0, \quad j = 1, 2, \dots, N \quad (58)$$

Where the system matrices are defined as:

$$M_{ji} = \int_0^L \mu(x) \phi_i(x) \phi_j(x) dx \quad (59)$$

$$R_{ji} = \int_0^L b \phi_i(x) \phi_j(x) dx \quad (60)$$

$$K_{ji} = \int_0^L T(x, t) \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} dx \quad (61)$$

This yields the matrix form:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{R}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0}$$

which can be solved numerically for $\mathbf{U}(t)$ using suitable time integration methods such as Newmark method. The matrices are :

$$M = \frac{\Delta x}{12} \begin{bmatrix} \mu_0 + 4\mu_1 + 3\mu_2 & 3\mu_0 - \mu_1 & 0 & 0 & \dots & 0 \\ \mu_0 - \mu_1 & \mu_1 + 4\mu_2 + 3\mu_3 & 3\mu_2 - \mu_3 & 0 & \dots & 0 \\ 3\mu_2 - 3\mu_3 & \mu_2 + 4\mu_3 + 3\mu_4 & 3\mu_3 - \mu_4 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots \\ 3\mu_{N-1} - \mu_N & 0 & 0 & 0 & \dots & \mu_{N-1} + 4\mu_N + 3\mu_{N+1} \end{bmatrix}$$

$$R = \frac{b\Delta x}{3} \begin{bmatrix} 2 & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & \dots \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} & \dots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{2} & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{2T_c\Delta x}{1} - 2\alpha + \alpha \left[\sqrt{1 + \left(\frac{U_1(t)}{\Delta x}\right)^2} + \sqrt{1 + \left(\frac{U_2(t)-U_1(t)}{\Delta x}\right)^2} \right] & -\frac{T_c\Delta x}{1} + \alpha - \alpha \sqrt{1 + \left(\frac{U_2(t)-U_1(t)}{\Delta x}\right)^2} & 0 & \dots \\ -\frac{T_c\Delta x}{1} + \alpha - \alpha \sqrt{1 + \left(\frac{U_2(t)-U_1(t)}{\Delta x}\right)^2} & \frac{2T_c\Delta x}{1} - 2\alpha + \alpha \left[\sqrt{1 + \left(\frac{U_2(t)-U_1(t)}{\Delta x}\right)^2} + \sqrt{1 + \left(\frac{U_3(t)-U_2(t)}{\Delta x}\right)^2} \right] & -\frac{T_c\Delta x}{1} + \alpha - \alpha \sqrt{1 + \left(\frac{U_3(t)-U_2(t)}{\Delta x}\right)^2} & \dots \\ 0 & \dots & \dots & \dots \end{bmatrix}$$

Time Evolution Using Newmark Method

The time evolution will be treated using the Newmark numerical method, as it is specifically designed for second-order differential equations, exactly like those encountered in mechanics.

The method is as follows: initially, an estimate is made for displacement and velocity using the displacement, velocity, and acceleration data from previous steps :

$$\begin{cases} U^{n+1} = U^n + \dot{U}^n \Delta t + \frac{1}{2} \left[(1 - 2\beta) \ddot{U}^n + 2\beta \ddot{U}^{n+1} \right] (\Delta t)^2 \\ \dot{U}^{n+1} = \dot{U}^n + \left[(1 - \gamma) \ddot{U}^n + \gamma \ddot{U}^{n+1} \right] \Delta t \end{cases} \quad (62)$$

Parameters beta and gamma determine accuracy and stability of the method. Having the previous values, the new displacement is calculated by the following equation:

$$M\ddot{U}^{n+1} + R\dot{U}^{n+1} + KU^{n+1} = 0 \quad (63)$$

Substituting the velocity \dot{U}^{n+1} :

$$M\ddot{U}^{n+1} + R\left(\dot{U}^n + \left[(1-\gamma)\ddot{U}^n + \gamma\ddot{U}^{n+1}\right]\Delta t\right) + KU^{n+1} = 0 \quad (64)$$

which leads to :

$$(M + \gamma\Delta tR)\ddot{U}^{n+1} + R\dot{U}^n + (1-\gamma)\Delta tR\ddot{U}^n + KU^{n+1} = 0 \quad (65)$$

and rearranged as :

$$(M + \gamma\Delta tR)\ddot{U}^{n+1} = -R\dot{U}^n - (1-\gamma)\Delta tR\ddot{U}^n - KU^{n+1} \quad (66)$$

Substituting \ddot{U}^{n+1} into the displacement equation yields:

$$\begin{aligned} (M + \gamma\Delta tR)U^{n+1} &= (M + \gamma\Delta tR)U^n + (M + \gamma\Delta tR)\dot{U}^n\Delta t + \frac{1}{2}(1-2\beta)(M + \gamma\Delta tR)\ddot{U}^n(\Delta t)^2 \\ &+ \beta(\Delta t)^2 \left[-R\dot{U}^n - (1-\gamma)\Delta tR\ddot{U}^n - KU^{n+1}\right] \end{aligned} \quad (67)$$

which can be rearranged to

$$\left[Q + \beta(\Delta t)^2K\right]U^{n+1} = QU^n + \Delta t\left[Q - \beta(\Delta t)^2R\right]\dot{U}^n + (\Delta t)^2\left[\frac{1}{2}(1-2\beta)Q - \beta\Delta t(1-\gamma)R\right]\ddot{U}^n \quad (68)$$

where $Q = M + \gamma\Delta tR$.

Defining

$$F(U^{n+1}) = \left[Q + \beta(\Delta t)^2K\right]U^{n+1} - T(U^n, \dot{U}^n, \ddot{U}^n) = 0 \quad (69)$$

To solve this nonlinear system of size $N \times N$, the Newton-Raphson method is used. The first step is to compute the Jacobian matrix of the system:

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial U_1} & \frac{\partial F_1}{\partial U_2} & \dots & \frac{\partial F_1}{\partial U_N} \\ \frac{\partial F_2}{\partial U_1} & \frac{\partial F_2}{\partial U_2} & \dots & \frac{\partial F_2}{\partial U_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_N}{\partial U_1} & \frac{\partial F_N}{\partial U_2} & \dots & \frac{\partial F_N}{\partial U_N} \end{bmatrix} \quad (70)$$

The partial derivatives are calculated as

$$\frac{\partial F_j}{\partial U_{n+1,w}} = Q_{jw} + \beta(\Delta t)^2K_{jw} + \beta(\Delta t)^2 \sum_i \frac{\partial K_{ji}}{\partial U_{n+1,w}} U_{n+1,i} \quad (71)$$

Finally we arrive at this :

$$\frac{\partial F_j}{\partial U_{n+1,w}} = \begin{cases} [Q_{w-1,w} + \beta(\Delta t)^2 K_{w-1,w}] \delta_{1,w-1} + [Q_{w,w} + \beta(\Delta t)^2 K_{w,w}] \delta_{1,w} + [Q_{w+1,w} + \beta(\Delta t)^2 K_{w+1,w}] \delta_{1,w+1} \\ + \alpha \beta \left(\frac{\Delta t}{\Delta x} \right)^2 \left(\left[\frac{U_1}{\sqrt{1 + \left(\frac{U_1}{\Delta x} \right)^2}} - \frac{2(U_2 - U_1)}{\sqrt{1 + \left(\frac{U_2 - U_1}{\Delta x} \right)^2}} \right] \delta_{1,w} + \frac{2(U_2 - U_1)}{\sqrt{1 + \left(\frac{U_2 - U_1}{\Delta x} \right)^2}} \delta_{2,w} \right) U_{n+1,1}, & j = 1 \\ [Q_{w-1,w} + \beta(\Delta t)^2 K_{w-1,w}] \delta_{j,w-1} + [Q_{w,w} + \beta(\Delta t)^2 K_{w,w}] \delta_{j,w} + [Q_{w+1,w} + \beta(\Delta t)^2 K_{w+1,w}] \delta_{j,w+1} \\ + 2\alpha \beta \left(\frac{\Delta t}{\Delta x} \right)^2 \left(-\frac{U_{w+1} - U_w}{\sqrt{1 + \left(\frac{U_{w+1} - U_w}{\Delta x} \right)^2}} \delta_{j-1,w} + \left[\frac{U_w - U_{w-1}}{\sqrt{1 + \left(\frac{U_w - U_{w-1}}{\Delta x} \right)^2}} - \frac{U_{w+1} - U_w}{\sqrt{1 + \left(\frac{U_{w+1} - U_w}{\Delta x} \right)^2}} \right] \delta_{j,w} + \frac{U_w - U_{w-1}}{\sqrt{1 + \left(\frac{U_w - U_{w-1}}{\Delta x} \right)^2}} \delta_{j+1,w} \right) U_{n+1,j}, & 1 < j < N \\ [Q_{w-1,w} + \beta(\Delta t)^2 K_{w-1,w}] \delta_{N,w-1} + [Q_{w,w} + \beta(\Delta t)^2 K_{w,w}] \delta_{N,w} + [Q_{w+1,w} + \beta(\Delta t)^2 K_{w+1,w}] \delta_{N,w+1} \\ + \alpha \beta \left(\frac{\Delta t}{\Delta x} \right)^2 \left(-\frac{2(U_N - U_{N-1})}{\sqrt{1 + \left(\frac{U_N - U_{N-1}}{\Delta x} \right)^2}} \delta_{N-1,w} + \left[\frac{2(U_N - U_{N-1})}{\sqrt{1 + \left(\frac{U_N - U_{N-1}}{\Delta x} \right)^2}} + \frac{U_N}{\sqrt{1 + \left(\frac{U_N}{\Delta x} \right)^2}} \right] \delta_{N,w} \right) U_{n+1,N}, & j = N \end{cases}$$

Then the update step is :

$$J(U^{n+1|k}) \Delta U^{n+1|k} = -F(U^{n+1|k}) \quad (72)$$

$$U^{(k+1)|n+1} = U^{n+1|k} + \Delta U^{n+1} \quad (73)$$

After a sufficient number of iterations, having the new displacement, the acceleration is updated as :

$$\ddot{U}^{n+1} = \frac{1}{\beta(\Delta t)^2} \left[U^{n+1} - U^n - \dot{U}^n \Delta t - \frac{1}{2}(1 - 2\beta)(\Delta t)^2 \ddot{U}^n \right] \quad (74)$$

and the new velocity as :

$$\dot{U}^{n+1} = \dot{U}^n + \left[(1 - \gamma) \ddot{U}^n + \gamma \ddot{U}^{n+1} \right] \Delta t \quad (75)$$

—
To produce sound, we assume sampling at frequency $f_s = 44.1$ kHz. For safety, the time step is chosen half as large:

$$\Delta t = \frac{1}{2f_s} \approx 10 \mu s \quad (76)$$

The spatial discretization step is chosen as:

$$\Delta x = \frac{L_{\min}}{100} = \frac{\lambda_{\min}}{200} = \frac{c}{200f_{\max}} = 2.5 \times 10^{-7} \sqrt{\frac{T_c}{\rho}} \quad (77)$$

Assuming $\rho = 0.0005$ kg/m and $T_c = 70$ N, it follows that

$$\Delta x \leq 2.5 \times 10^{-4} \quad (78)$$

Finally, because the string is fixed at both ends, the time-domain signal is recovered as:

$$s(t) = \int_0^L y(x, t) \sin\left(\frac{n\pi x}{L}\right) dx \quad (79)$$

Conclusions:

The antinodes in a standing wave contain only kinetic energy. Conversely, the nodes have only potential energy.

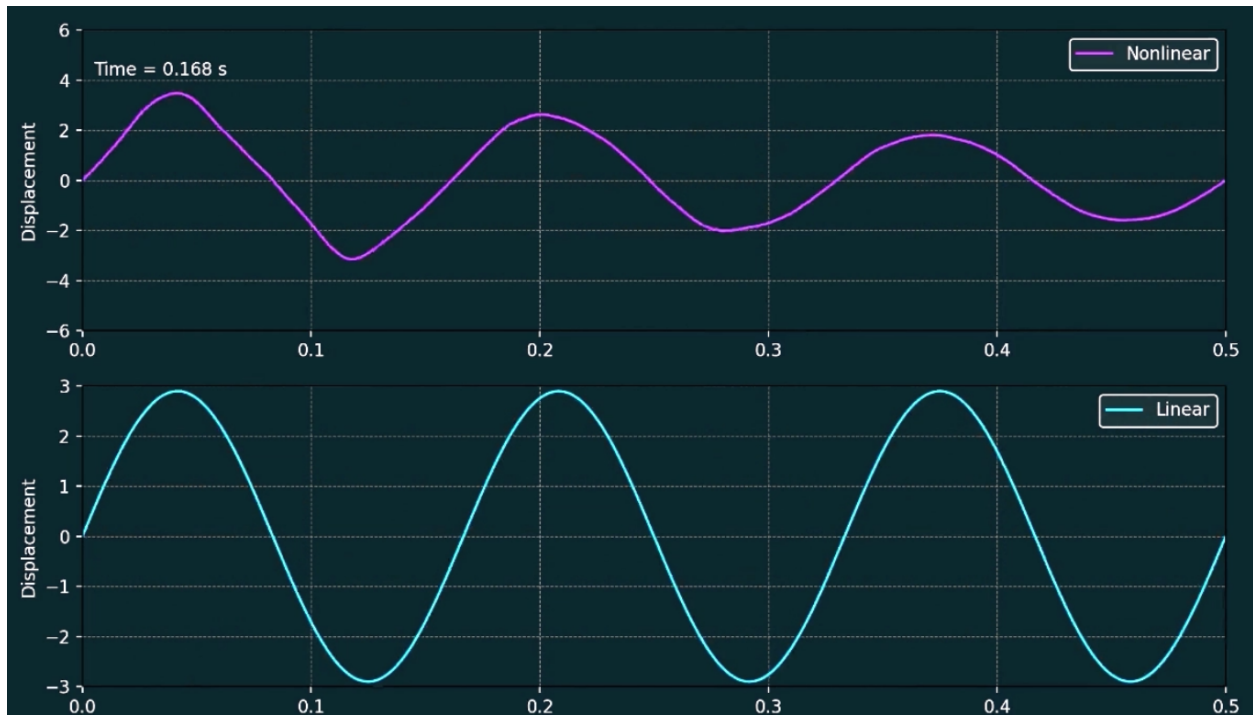


Figure 2: Linear and non-linear vibration of the string from $x = 0$, to $x = \text{wavelength}/2$

When the antinodes pass through the equilibrium position, they have maximum kinetic energy, while at maximum displacement they have zero kinetic energy. At that point, the nodes have maximum potential energy due to the deformation of the string around them.

Energy is transferred from the antinodes to the nodes and vice versa during the period of a standing wave, but it does not propagate beyond these points. So, there is instantaneous energy transfer, but its average value over one period is zero, resulting in no net energy transfer.

The linear approximation is valid only as long as the following condition holds:

$$A \leq \lambda \quad \text{or equivalently} \quad A \leq \frac{c}{\omega}$$

This condition may fail in two characteristic cases:

- When we have very large amplitudes, even with low frequency, which can occur due to interference.
- When high-frequency waves are present.

In such cases, the tension along the string (restoring force due to curvature) is no longer constant, as the elongation increases it due to the elasticity of real strings.

These phenomena have several interesting consequences, such as:

- Wave steepening and spreading due to dispersion,
- Creation of asymmetries,

- Energy transfer between different modes,
- Generation of harmonics, etc.

Finally, it is worth noting that near the peaks and troughs, the curvature becomes large, and thus some potential energy is stored, since real strings have cross-section and resist bending. Of course, due to the small cross-section, this resistance is weak.